How to choose a natural number at random?

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The foundations of the theory of probability have seen a significant evolution during the 20th century, through the developments of thinkers like von Mises, Kolmogorov and de Finetti to name but a few. However a lot of important problems still remain open and receive wide attention today from philosophers, as well as from mathematicians and economists. One of these problems is whether the axiom of countable additivity (henceforth referred to as CA) should be included amongst the other more accepted axioms. This question will be the concern of this text, albeit posed in a strictly less general form; namely whether this axiom should be included amongst the axioms if one accepts the subjective foundations of probability.[1] I shall approach the problem by restricting my attention to the experiment of 'choosing a natural number at random', as this has been at the centre of the debate. Several arguments on both sides of the debate will be analyzed, which shall lead us to propose an interpretation of the experiment that is consistent with CA. From this we shall conclude that CA is an acceptable axiom.

1. The dilemma.

The subjective interpretation of probability was developed almost single-handedly by de Finetti, so it makes good sense to commence our investigation with his viewpoint on the matter. After Kolmogorov had axiomatized probability theory with the aid of measure-theory as developed by Borel and Lebesgue, de Finetti sought to interpret these axioms in a personalistic manner.[2] In doing so he was led by the guiding principle of coherence : "it would be inconsistent to consider fair a bet that produces (*uniformly*) a loss in any case." (de Finetti [1972], p. 84)[3] This principle assures the axiom of finite additivity (the sum of the probabilities of two disjoint events is equal to the probability of the union of those events)[4], but it seems to leave open the question of CA:

For a countable set of disjoint events $E_1, E_2, E_3, \ldots, E_n, \ldots$

$$P(\bigcup^{\infty}_{i=1} E_i) = \sum^{\infty}_{i=1} P(E_i).$$

1 The acceptance of CA is less controversial in objective interpretations, in fact on the frequentist interpretation it is simply untrue.
2 More on the history of probability theory can be found in von Plato (1994).
3 De Finetti uses the term consistency.
4 For a proof of this see de Finetti (1972), p. 77-79.

Kolmogorov accepted the axiom of CA without giving any justification for it, since he did not consider it an essential statement. The reason for this lies in his finitist background:

> Kolmogorov's reason for assuming denumerable additivity was that it would be a harmless idealization. If the probability *P(E)* of a finitary event *E* can be derived with the help of probabilities of infinitary events, involving computations by the rule of denumerable additivity, there would also be a derivation without them. Following Hilbert, Kolmogorov thought the infinite is an extension of properly finitist mathematics, such that the above property of derivations of results with a finitary meaning holds. Infinite events would work as 'ideal elements' in probability theory, and their use would not extend the class of provable truths about the finite. That, of course, turned out to be a mistaken idea after Gödel showed that Hilbert's programs for saving the consistency of mathematics cannot be carried through with finitary means. (von Plato [1994], p. 277).

De Finetti on the contrary realized that the addition of CA could not be taken for granted or seen merely as a convention, since it conflicts with another important intuition regarding probability functions. It is in this context that he mentions the experiment of choosing a natural number at random.

> By definition, this is a random quantity *X* which takes any given one of the possible values $X=1, 2, 3, \ldots, n, \ldots$ with probability zero; [...]. In this case *complete additivity does not hold* because the elementary cases have each zero probability: $p1 = p2 = \ldots = pn = \ldots = 0$, so that their sum is again zero, while the probability of the union, which is the sure event, is equal to one. (de Finetti [1972], p. 86).

The reason why CA cannot hold in this example is that the notion of 'choosing at random' immediately brings up the idea of a *uniform distribution*, since we may consider each event to be as likely as any other. If we see no reason to discriminate between all possible outcomes - we might not have any information which would lead us to do so - it makes sense to consider all events equiprobable. However since there are countable many outcomes, any probability assignment that assigns a strictly positive probability to each event would violate the requirement that a probability may never exceed one, forcing us to assign to each event probability zero. The problem we are left with is to choose sides in the following dilemma:

> *either to exclude problems of the type of the integer chosen at random (and then give reasons why such problems should not be considered), or else, to forego conclusions based on complete additivity*

(and then be ready to accept some technical complications) (de Finetti [1972], p. 87).

Essentially two kinds of arguments have been pervasive in the debate on CA: arguments revolving around the legitimacy of uniform distributions on a countably infinite partition, and Dutch book arguments for CA. Other arguments seem to have had less influence because they focus on certain mathematical consequences (such as the strong law of large numbers, or the conglomerability paradox) of accepting CA as an axiom. These can never really be conclusive since no matter how great the benefits or disadvantages of the axiom, the question remains open whether the axiom itself is forced upon us by the conditions of consistency relevant to the subjectivist theory. Therefore we will limit our attention to the first two kinds of arguments, since they focus on conceptual issues which are at the core of the foundations of subjective probability theory.[5] In the following section we will consider Dutch book arguments for CA.

2. A Dutch book for CA: an appeal to our intuition.

Dutch book arguments are often seen as decisive when dealing with foundational issues. The reason for this is that the fundamental idea of coherence is equivalent to the impossibility of having a Dutch book made against you. In fact de Finetti gives a Dutch book argument to establish the axiom of finite additivity (de Finetti [1937], p. 103-104). Consequently it is not surprising that many authors have tried to devise Dutch book arguments in favour of CA. A well-known example of such an attempt is Williamson's (Williamson [1999]). He considers it to be a knock-down argument in favour of CA, and other authors have agreed on the importance of this kind of argument.[6] What is surprising is that Williamson ignores the fact that de Finetti himself has given a Dutch book for CA, although his argument is nothing but a more formal and technical formulation of de Finetti's original. Technicalities should be avoided when they obscure the essential matter rather than place an argument on firmer ground. For our purposes it will suffice to have a look at Bartha's version of Williamson's argument:

> Suppose that pi is our fair betting quotient for the proposition that ticket *i* wins, and 1 is our fair betting quotient for the proposition that some ticket wins. Suppose that countable additivity is violated, so that
>
> $$p1 + p2 + ... < 1.$$
>
> Each of the following bets is fair: bet *against* ticket *i* with a stake of $1 and betting quotient pi. This

5 This reason is also stated by both de Finetti (1972, p. 83) and Williamson: “Thus, in a sense, arguments from the consequences of countable additivity are irrelevant to whether countable additivity is an appropriate axiom for subjective probability.” (1999, p. 404).

6 See for example Bartha (2004) and Howson and Urbach (1993).

bet pays pi dollars if ticket *i* loses (we win our bet), and (pi – 1) dollars if ticket *i* wins (we lose our bet). The system consisting of all these bets taken simultaneously is fair. Suppose now that ticket N wins, as must happen for some N. We win pi for all tickets other than N, and pN – 1 for ticket N. Our net gain is therefore

$$(p1 + p2 + \dots) - 1$$

which, by assumption, is negative. So no matter what happens, we lose money. This constitutes a Dutch Book. (Bartha [2004], p. 306).

De Finetti gives essentially the same argument, with the difference that the stakes are reversed so that you have what one could call an opposite Dutch book: a system of bets so that no matter what happens the net gain is positive. "if the sum of probabilities *pn* is $p < 1$, it would be possible, by entering the infinite number of available bets, to receive 1 in any event for a total payment of amount *p*, and this is clearly unreasonable." (de Finetti [1972], p. 91)

For some this might seem the end of the matter, but de Finetti rejects the argument on grounds of being circular. He continues where we left him: "But in reality the argument is circular, for only if we know that complete additivity holds can we think of extending the notion of combinations of fair bets to combinations of an *infinite* number of bets, with the corresponding sequence of betting odds." In the case of a finite number of bets, we simply state that the expected payoff should be zero if one bets on every possible event and at the same time bets against every possible event. This move is justified because postulating that the expected payoff is zero is nothing more than an explanation of what we mean by the idea of rational coherence in relation to a system of fair bets. De Finetti's argument would then seem to be directed at the presupposition that a similar move is justified in case of an expected payoff when considering a system of an infinite number of bets, whereas we lack a similar intuition concerning coherence.

Contrary to this Spielman claims that we *do* have such an intuition. "De Finetti is mistaken. Countable additivity is not directly presupposed. We simply set down as a postulate an intuition about fairness which virtually everyone would share." (Spielman [1977], p. 256) However I do not see how an intuition about fairness, considered in the context of a countable number of bets, could do without countable additivity. This is not to say that I disagree on there being such an intuition; rather I think countable additivity is part of this intuition. The situation is analogous in this respect to the 19^{th} century problem of giving proofs for the principle of mathematical induction. Frege assumed he had given such a proof, but Poincaré convincingly argued that such a proof would always be circular: at one point in the proof one will have to leap from the finite to the infinite, hence applying induction. The intellectually honest thing to do is take the principle at face value and treat it as a basic intuition.

Although de Finetti would probably agree that, intuitively, CA is a straightforward property, his claim would likely be that unfortunately in this case we cannot rely on intuition alone. If we do, we will fall prey to contradictions, as in the random integer example. The contradictions are due to another equally plausible intuition which conflicts with CA, namely that a rational agent has good reason to assign a uniform distribution over a countable partition in the setting of the random integer example. Dutch book arguments alone can do no more than appeal to the intuitive attractiveness of CA. In order to make it an acceptable postulate, we must examine this second intuition from which the conflict originates.

3. Is there a case for a uniform distribution over a countable set?

We return to our example of choosing a random integer.[7] As noted earlier, we may not have any reason for regarding some integers to be more likely than others. Hence it seems natural that one should be allowed to adopt a uniform distribution over the integers. But does such a distribution make any sense? Could one imagine a mechanism for choosing an integer at random which would bring it about? The question pertains to the interpretation of the vague notion of 'choosing at random' when applied to infinite collections. Spielman criticizes the possibility of adopting a uniform distribution on exactly these grounds:

> The phrase 'integer chosen at random' is far from clear. I have not the slightest idea of how to go about selecting an integer from the class of all integers in such a way that each number has an equal chance of being selected. [...] It is not that I am unwilling to consider purely conceptual experiments. On the contrary, I can think of all sorts of 'chance mechanisms' for selecting integers from the set of all integers. [...] But all such methods yield asymmetrical probabilities... (Spielman [1977], p. 254)

Granted that indeed no mechanism exists which yield a uniform distribution over the set of all integers (so far, no one has doubted this assumption; with good reason, I believe), can a person be considered rational if his dispositions to place bets reflect the existence of such a distribution?

According to Williamson he can.

> The point at issue is not the viability of some objective mechanism but rather an agent's epistemic state. [...] The agent may even believe that no mechanism for picking a number according to an

7 Note that before we spoke of a random *natural number*, we will alternate between the two options, since both experiments are the same.

objective uniform distribution exists and that the number is practically certain to be in some finite subset, but not know which subset, and so again be predisposed to place bets according to a uniform distribution. (Williamson [1999], p. 407).

This counter-argument sounds convincing, but upon closer examination it falls apart. Each finite subset which the agent thinks of corresponds to a certain physical mechanism for selecting an integer. Implicit in the argument is an application of the principle of indifference[8] : because the agent has no idea which mechanism will actually be used, all mechanisms supposedly being sufficiently like each other according to relevant criteria, he considers them equiprobable. The combination of the uniform probability distribution on the mechanisms and the specific finite distributions of each mechanism would then presumably lead to a uniform distribution on the integers. Although this reasoning is not stated explicitly in the argument, I see no other way for the agent to have arrived at his conclusion.

The assumption that all mechanisms are sufficiently like each other is a necessary precondition to the application of the principle of indifference, however it is clear that one can think of countless examples of mechanisms which are extremely unlike each other and are hardly comparable at all. Imagine, for example, picking up a phone book and counting the number of names in it, or taking the absolute value of the integer closest to the tenth zero of some twelfth degree algebraic equation, or simply writing down the first number one thinks of.

For the sake of argument it could be granted that such considerations would complicate the agent's reasoning in such a way that he could not arrive at a conclusion at all, and he may therefore abstract away from the actual mechanisms and simplify the number of possibilities significantly. Instead of thinking in terms of mechanisms, he considers every finite subset of the set of all integers to be the representative of a group of mechanisms for choosing an integer. Even if this were granted to salvage the argument, it would still fail.

First of all, all finite subsets of integers are not at all sufficiently like each other to be considered as events which are alike; for their size is the most relevant criterion of comparison and it varies indefinitely, stretching at least from one to any other number taken as large as one pleases. Secondly, in order to arrive at the uniform distribution over the integers, one has to combine the distribution of all the subsets with each of the distributions which appears in each subset. In order for this combination to yield a uniform distribution over all integers, we would again have to apply the principle of indifference to argue that we may assume a uniform distribution as the average of

8 The principle of indifference states that if there are n incompatible and exhaustive possibilities which are indistinguishable except for their names, then each possibility should be assigned a probability equal to $1/n$.

the distributions of all mechanisms in each particular group. Even more, we would have to apply the principle yet another time to defend the assumption that each group will contain the same amount of members. However this application would be very hard to defend, because it is clear that one can come up with significantly more mechanisms which allow a small number of outcomes than mechanisms which have as many outcomes as there are atoms in the universe. Thirdly, the restriction to finite distributions is unwarranted, since there are numerous mechanisms which yield a distribution over all integers without assigning any particular integer probability zero.[9]

Thus the counter-argument as it is stated by Williamson is insufficient to substantiate the claim that it may be reasonable to adopt a uniform distribution over the integers. It could be objected that there may be some other way of comparing all possible mechanisms that *does* give a uniform distribution as the weighted average of all mechanisms. We cannot exclude the possibility that such a comparison exists; what we can do is make this seem more unlikely by attacking candidates for such a comparison. So far the only candidate is the one which has been put forward by Bartha; who gives an argument to support the claim Williamson makes. In it he explicitly states a method of comparing all mechanisms that leads to a uniform distribution as the weighted average.

> For every mechanism *M* that favours integer *m* over *n*, there will be another mechanism *M '* that is identical except that the values P(m) and P(n) are exchanged. (To construct *M '* , just do a swap and then apply *M*.) The full set-up of the lottery [the experiment of choosing a random integer] includes the choice of a mechanism. So our subjective probability assignments should ensure that P(m) = P(n) for any *m* and *n*. The fact that the particular mechanism selected is inevitably non-uniform does not force us into non-uniform subjective probabilities... (Bartha, 2004, p. 305).

Bartha does not mention any restrictions which limit the form the probability distribution of a specific mechanism may take, rather the argument is framed in a purely formal setting and abstracts away from any physical considerations which would disallow certain distributions. At this level of generality, it may therefore be assumed that a mechanism is simply a bijection between the integers and a countable set of fractions with the property that when they're added together their sum is equal to one.[10] The set of all such bijections would then be the collection over which a uniform distribution may be assumed. Indeed, at first sight I do not see any reason why an agent should be

9 An example would be a sequence of toin cosses where the number of consecutive tosses that gave heads is the integer chosen and the sequence is broken off once the coin lands tails.

10 Of course this property is equivalent to CA , so it might be objected that the argument interpreted in this way is circular. However the sum being equal to one is irrelevant, the only restriction is that it may not exceed one. The question whether it should always equal one or not does not make any difference.

called irrational if he formalizes the experiment in this way, having no information whatsoever on the actual physical restrictions which might exist on the form probability distributions of actual physical mechanisms can take. Of course there exist only a finite amount of realizable mechanisms, but since we don't know anything about their appearance the idea of an infinite amount of them is a justifiable generalization.

On the other hand, the idea of swapping *m* and *n* – which is at the crux of the argument – implies that the set of all mechanisms should be at most countable. To see this, note that the final distribution we end up with is the weighted average of all the distributions at hand; but the concept of a weighted average makes sense only in a countable context. We will use a diagonal argument to show that in fact the above mentioned set of bijections is non-denumerable, thereby discrediting Bartha's argument.

Assume we have a bijection between the natural numbers and the set of all mechanisms. Let *Mn* stand for the n-th mechanism in the list, and let *MnPs* stand for the probability *Mn* assigns to the integer *s*. We can construct a 'diagonal mechanism' *DM* as follows: If $M1P1 \neq 1/2$, then set $DMP1 = 1/2$, if even further $M2P2 \neq 1/4$, then set $DMP2 = 1/4$, and so on. If at a certain moment the antecedent is untrue, so that for a certain integer *a* $MaPa = 1/(2^a)$, we set $DMPa = 0$. In fact we will set $DMPn = 0$ for all values $\geq a$, until we find a number *b* for which $MbPb \neq 1/(2^a)$. The value $1/(2^a)$ will then be inserted in *DM* at the integer *b*, so that $DMPb = 1/(2^a)$. [11] We then proceed to the next integer *c* for which $McPc \neq 1/(2^{(a+1)})$ and set $DMc = 1/(2^{(a+1)})$, having again set all values of *DMn* in between *b* and *c* zero. By continuing this process we end up with a mechanism *DM* which is different from each *Mn*, since $DMn \neq MnPn$. *DM* satisfies the requirement that its probability assignments add up to one, since

$$\sum_{i=1}^{\infty} DMi = \sum_{i=1}^{\infty} 1/(2^i) = 1.$$

As with Cantor's original diagonal argument, we can conclude that the assumed bijection cannot exist. Therefore the amount of mechanisms is non-denumerable.[12]

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11 Here we have assumed the existence of such a *b*. A proof of this goes as follows. If such a number *b* would not exist, then from *Ma* onwards all mechanisms assign the value $1/(2^a)$ to some integer. This means that there is only a finite amount of mechanisms (namely at most *a* of them) which do not assign any integer the probability $1/(2^a)$. This is untrue.

12 Of course this is an absurd conclusion; the reason why it could be reached was that our formalization of the notion of a mechanism was far too general. In fact it allows there to be “random mechanisms”, a doubtful concept. However this level of generality was required for Bartha's argument to get of the ground in the first place. For if we were to interpret the correspondence between mechanisms and distributions literally - as opposed to interpreting it as an idealization -, the swapping argument would not work either. For every mechanism *M '* that swaps *m* and *n*, there exists a mechanism that swaps *m* and *n* again, and again if one wishes, continuing indefinitely. Can all these mechanisms taken together be said to cancel each other? The situation is analogous to that of the outcome of adding all terms in the sequence: 1, -1, 1, -1, ...If one sums the terms in this order (1 + -1) + (1 + -1) +..; one gets zero, if one sums them 1 + (-1 + 1) + ... one gets one. It is because of this kind of indeterminacy that rules were agreed upon for taking limits of sequences, however in the case of mechanisms such rules have evidently not yet been agreed

Let's resume where we stand. Spielman raised the objection that one should not assign a uniform distribution to the integers since we cannot imagine any mechanism yielding such a distribution. Williamson and Bartha have tried to argue that despite the fact that such a mechanism is inconceivable, it may nonetheless be rational to be predisposed to such a uniform distribution. We have tried to show that they have failed to do so. This brings us to the question: what can be considered rational for a person to believe when confronted with the 'random integer' example? Our purpose is now to give various interpretations of such an experiment which allow this question to be answered.

4. Some models of random choice.

If probability is viewed from the perspective of measure theory, it is reduced to a purely mathematical notion. It is a real-valued function defined on a class of sets, this being subject to certain limitations. The semantical aspect of probability enters the scene when one wants to determine a correspondence between a certain mathematical model and an actual physical set-up of which it ought to be a model. Usually this correspondence is deficient, in the sense that the model is an idealized situation of what actually happens. For example, the events in the universe of the experiment of choosing a random real number in the unit interval can be seen as idealizations of the result of an infinite sequence of throws of a ten–sided dice. A uniform distribution over some finite set can be seen as an idealization of the experiment of randomly picking a marble out of an urn. Furthermore, this correspondence will be one-to-many: one mathematical model can represent different chance experiments, but not vice versa. The objective interpretation of probability holds that only one model can capture the probability distribution of a physical event.

Subjective probability adds an extra degree of freedom: there may be different models (which, in this perspective, are sets of degrees of belief) which a rational agent can perceive as the representation of an actual event. The only requirement is that his beliefs are coherent, meaning no Dutch book can be made against them. Although at first sight this extra freedom appears only to create new connections between models and experiments, it also forbids certain connections which in an objective interpretation make perfect sense. Examples of the latter are cases in which the experiment is purely conceptual, by which I mean a thought-experiment to which no actual experiment can correspond, be it merely as an approximation. For objective probability all that is needed is a connection between a model and an experiment, regardless of whether it is conceptual or actual. Subjective probability requires that we are able to imagine an agent placing bets on the

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upon. Some other correspondence between mechanisms and sequences may be possible to save the argument, but it would have to be very far-fetched.

possible outcomes of the experiment. An agent susceptible to a Dutch book is considered irrational because it is considered irrational to voluntarily place yourself in a situation in which you will lose money no matter what happens. Therefore it is essential that a probability experiment is in some form or another a representation of an actual situation in which we can imagine an agent to be placed. The previous paragraph will become clearer when we illustrate it with the 'random integer' example.

Considered as a thought-experiment, I can come up with three interpretations of the phrase 'an integer was chosen at random'. We might imagine it to be a generalisation of an earlier example, that of an urn containing numbered marbles. This interpretation depends on the idea of an actual infinite: the urn contains infinitely many marbles at the same time. God chooses a marble at random out of the urn. Since Cantors actual infinities have become well-established mathematical objects there is no a priori reason to banish them from objective probability theory. Whether one wants to accept such experiments or not depends on whether they are mathematically interesting. The matter is different for subjective probability. In this interpretation the experiment does not represent a situation in which a person might purchase bets on the possible outcomes of some actual experiment. Therefore probability theory cannot be applied to it.

A second interpretation is that of a potentially infinite sequence of coin tosses. If the first toss gives heads, the number will be even, otherwise it will be odd. After this toss we renumber the remaining integers, solely as a matter of convenience, so that after the second toss we can again eliminate either the even or the odd numbers. We continue this process toss after toss. For example if the first three tosses gave heads, tails, heads, the number to be chosen will be an element of the set {6, 14, 22, 30..., 6 + 8*n, ...}. Although after each toss we are still left with an infinite amount of possibilities, an infinite amount of numbers will have been eliminated as well. After each toss the remaining integers will become larger and larger, growing to infinity. In fact if we allow actual infinity, meaning that we allow a countable infinite sequence to have taken place, the outcome will be nothing but omega.[13] Of course if we do this we are faced with the same objection as in the first interpretation, so we will have to interpret the sequence as a potential infinite if we want it to be relevant for subjective probability.

On this second interpretation the thought-experiment *is* related to an actual experiment. The set up could consist of a machine which chooses a natural number at random. Before it starts working, bets may be placed on any number. Then the machine is switched on and starts eliminating numbers according to the above-mentioned process. After a finite amount of time all

13 Of course by interpreting the sequence as an actual infinite, omega has to be added to our event-space. It will then become the set X = {all integers U omega}.

tickets for bets on specific integers will have become worthless. However, if this experiment is to be in some way a representation of choosing a random number, infinity – which is nothing more than a symbol for the event that the machine will never stop - has to be added to our event-space as a possible outcome in order to avoid a contradiction. Otherwise a person may object that the experiment was a scam, since no number will ever be chosen. On the other hand if bets on infinity – and thus on the machine never stopping - are allowed, it can be said that the experiment is the closest one can get to choosing a random integer. From this we conclude that the only coherent probability distribution given this set-up is to assign to each finite number probability zero and probability one to infinity. [14]

The last interpretation is the most realistic one. Someone comes to you holding a piece of paper with an integer written on it and proposes you to bet on which number it is. After you have both placed all your bets, the piece of paper is turned around and money is exchanged.

This is the kind of set-up Williamson and Bartha were concerned with when giving their arguments for a belief in a uniform distribution. Although you know that the number was not chosen randomly, there are good reasons to have beliefs which are indistinguishable from those a person would mistakenly hold who does believe the number was chosen randomly. Combining the results of Sections 2 and 3, this line of reasoning has lost its appeal. On the basis of our intuitions leading to a Dutch book for CA, we would like to accept CA as an axiom. Yet in order to do this an obstacle has to be removed which comes in the form of an apparently strong intuition on the legitimacy of uniform distributions over countable partitions. Part of this removal has been accomplished by undermining the reasons a rational person would have to end up with such a distribution. It would be entirely removed if we could give an answer to the question posed at the end of Section 2: what *can* be considered rational for a person to belief when confronted with the 'random integer' example?

Up till now we have restricted ourselves to one side of a disjunction related to the random integer example. Either one opts for a uniform distribution, or one chooses a different distribution. After what has been said, it seems we will have to go with the second option. This is the solution Williamson proposes at the end of his article. A rational agent, knowing that a uniform distribution is not to be chosen, will simply choose some distribution based on entirely subjective grounds. “So any discrimination between the events will have to have a semantic basis which may not always be

14 A similar interpretation is given by Howard (Howard, 2006). However he discusses *only* this interpretation, entirely neglecting the more problematic third interpretation, and claims to have shown a way out of de Finetti's dilemma in having done so. This is a grave oversimplification.

available, ruling out the goal of the logical interpretation, a unique 'most rational' *bel* [belief distribution]. Subjectivity is here to stay." (Williamson [1999], p. 413). In the terminology of mechanisms this can be expressed by stating that a rational person will choose a mechanism based entirely on subjective grounds. Simply to state this is not enough, to be convincing we need a good reason why a rational person would do this. This would shift the weight over the balance on to the other scale: instead of there being good reasons to assign a uniform distributions, there is on the contrary a good reason to assign anything but a uniform distribution.

A person *A* is confronted with an unknown integer on a piece of paper written on it by *B*, and has to decide on how to place his bets. He knows the number was chosen via some biased mechanism. *A* could then ask himself: "How did *B* ever decide on which number to write down?" Assuming *B* wanted to make the bet as profitable as possible, he probably tried to choose the number in a very elaborate way. A plausible way of doing this is thinking of all kinds of strange mechanisms for choosing numbers, and then picking out one and applying it. Here it is important to note that at some stage or other his cleverness for coming up with strange mechanisms has to come to an end and he has to pick one out without any reason. This means he will have chosen a mechanism entirely on subjective grounds.[15] *A* could then go on to argue: "If I go with a uniform distribution, I can be absolutely certain that my choice will be in no way similar to *B*'s choice. If I choose another distribution, based on entirely subjective grounds, I may at least have some hope that human psychology is not that complex and my choice will in some ways be similar to *B*'s." Therefore I would say that *A* has a good reason not to choose a uniform distribution. Although we end up agreeing with Williamson on this matter, we have been led to this conclusion following a different – and at times an opposite - path.

5. Conclusion.

Whether CA is included amongst the axioms of subjective probability theory or not is an important question. Dutch book arguments for CA are indecisive to answer it, because we already intuitively assume the principle when allowing the 'infinite bets' context necessary for them. If nothing were to stand in the way of letting our intuition guide us in this assumption, CA could be accepted. Arguments on there being good reasons for assigning uniform distributions to countable sets *would* stand in the way if they were correct. We have tried to show that the ones given in the literature are incorrect. To resolve the matter entirely we had a look at some possibilities for interpreting the idea of 'choosing an integer at random' consistently. We found that none of them conflicted with CA.

15 These grounds may well be of a probabilistic nature, he could for example have thrown a dice. What these grounds are is completely irrelevant, otherwise we could not properly call them subjective.

Therefore I suggest that we add the axiom of countable additivity to subjective probability theory.

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